STRUCTURES AND OPERATORS ON ALMOST-HERMITIAN MANIFOLDS(1)

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Introduction. It is well known (see, for instance, [6, §9.2]) that for a Kählerian structure the real Laplace-Beltrami operator Δ commutes with real operators L and Λ defined in §3, and the complex Laplace-Beltrami operator $\Box$ is real and equal to $\frac{1}{2}\Delta$. The purpose of this paper is to study the converse of these three properties of Kählerian structures.

§§1 and 2 contain fundamental notations and definitions of various almost-Hermitian structures, as well as real and complex operators, on an almost-Hermitian manifold. In §3 we shall prove

THEOREM 3.1. *For an almost-Hermitian structure of dimension n $(\geqq 2)$, if Δ commutes with the operator L or Λ with respect to all forms of any degree p $(0 \leqq p \leqq n-2)$, then the structure is Kählerian.*

It is known that the commutativity of Δ with L or Λ plays a crucial role in the proof of the Hodge's well-known theorem concerning the relationship between the effective harmonic forms and Betti numbers of compact Kählerian manifolds. Thus Theorem 3.1 kills a possibility of extending the Hodge's theorem to more general manifolds. It should also be noted that recently Weil [7] used the commutativity of some operators to characterize Chern's generalization of Kählerian structures. From Weil's result and our Theorem 3.1 here it seems natural to characterize a structure by using the commutativity of some operators.

In §4 we study the realization of the complex operator $\Box$ by establishing the following theorems.

THEOREM 4.1. *The complex operator $\Box$ for an almost-Hermitian structure is real with respect to every form of degree 0 if and only if the structure is almost-semi-Kählerian. Moreover, with respect to every form of degree 0, if $\Box$ for an almost-Hermitian structure is real, then $\Box = \frac{1}{2}\Delta$.*

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THEOREM 4.2. *If for an almost-Hermitian structure the relation* $\square = \frac{1}{2}\Delta$ *holds for all forms of degrees* 0 *and* 1, *then the structure is Kählerian.*

Kodaira and Spencer [4] have shown that if the relation $\square = \frac{1}{2}\Delta$ holds for an almost-Hermitian structure, then the structure is integrable. Theorem 4.2 was a conjecture for some time, and was proved very recently by A. W. Adler [1] by a different method under a stronger assumption that the relation $\square = \frac{1}{2}\Delta$ holds for a Hermitian structure and all forms of degrees 0, 1 and 2.

For more general case we shall have

THEOREM 4.3. *For an almost-Hermitian structure, if the complex operator* $\square$ *is real with respect to all forms of degrees* 0 *and* 1, *then it is also with respect to all forms of degree* 2.

It seems that the conclusion of Theorem 4.3 could be extended to all forms of degree p (> 2). However, due to the complication of the calculation the author is unable to show it.

Throughout this paper, the dimension of a manifold M^n is understood to be n ($\geqq 2$), and all forms and structures are of class at least C^2.

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1. **Notations and real operators.** Let M^n be a Riemannian manifold of dimension n ($\geqq 2$), $\| g_{ij} \|$ with $g_{ij} = g_{ji}$ the matrix of the positive definite metric of the manifold M^n, and $\| g^{ij} \|$ the inverse matrix of $\| g_{ij} \|$. Throughout this paper all Latin indices take the values $1, \cdots, n$ unless stated otherwise. We shall follow the usual tensor convention that indices can be raised and lowered by using g^{ij} and g_{ij} respectively; and that when a Latin letter appears in any term as a subscript and superscript, it is understood that this letter is summed for all the values $1, \cdots, n$. Moreover, if we multiply, for example, the components a_{ij} of a tensor of type $(0,2)$ by the components b^{jk} of a tensor of type $(2,0)$, it will always be understood that j is to be summed.

Let $\mathfrak{N}$ be the set $\{1, \cdots, n\}$ of positive integers less than or equal to n, and $I(p)$ denote an ordered subset $\{i_1, \cdots, i_p\}$ of the set $\mathfrak{N}$ for $p \leqq n$. If the elements $i_1, \cdots, i_p$ are in the natural order, that is, if $i_1 < \cdots < i_p$, then the ordered set $I(p)$ is denoted by $I_0(p)$. Furthermore, denote the nondecreasingly ordered p-tuple having the same elements as $I(p)$ by $\langle I(p) \rangle$, and let $I(p; \hat{s} \,|\, j)$ be the ordered set $I(p)$ with the s-th element i_s replaced by another element j of $\mathfrak{N}$, which may or may not belong to $I(p)$. We shall use these notations for indices throughout this paper. When more than one set of indices is needed at one time, we may use other capital letters such as J, K, L, $\cdots$ in addition to I.

At first we define

$$(1.1)\quad \varepsilon^{J(p)}_{K(p)} = \begin{cases} 0, & \text{if } \langle J(p)\rangle \neq \langle K(p)\rangle, \\ 0, & \text{if } J(p) \text{ or } K(p) \text{ contains repeated integers,} \\ +1 \text{ or } -1, & \text{if the permutation taking } J(p) \text{ into } K(p) \text{ is even or odd.} \end{cases}$$

By counting the number of terms it is easy to verify that

$$(1.2)\qquad \varepsilon^{I(p)J(n-p)}_{1\cdots n}\varepsilon^{1\cdots n}_{I(p)K(n-p)} = p!\varepsilon^{J(n-p)}_{K(n-p)},$$

$$(1.3)\qquad \varepsilon^{I(p)J(q)}_{K(p+q)}\varepsilon^{L(p)}_{I(p)} = p!\,\varepsilon^{L(p)J(q)}_{K(p+q)}.$$

On the manifold M^n, let ∇ denote the covariant derivation with respect to the affine connection Γ, with components Γ^i_{jk} in local coordinates $x^1,\cdots,x^n$, of the Riemannian metric g, and let ϕ be a differential form of degree p given by

$$(1.4)\qquad \phi = \frac{1}{p!}\,\phi_{I(p)}dx^{I(p)} = \phi_{I_0(p)}dx^{I_0(p)}\ ,$$

where $\phi_{I(p)}$ is a skew-symmetric tensor of type $(0,p)$, and we have placed

$$(1.5)\qquad dx^{I(p)} = dx^{i_1}\wedge\cdots\wedge dx^{i_p}.$$

Then we have

$$(1.6)\qquad d\phi = (d\phi)_{I_0(p+1)}dx^{I_0(p+1)},$$

where

$$(1.7)\qquad (d\phi)_{I(p+1)} = \frac{1}{p!}\,\varepsilon^{kJ(p)}_{I(p+1)}\nabla_k\phi_{J(p)}.$$

Denote

$$(1.8)\qquad e_{I(n)} = \varepsilon^{1\cdots n}_{I(n)}(\det(g_{ij}))^{1/2}.$$

Then by using orthonormal local coordinates $x^1,\cdots,x^n$ and the relation (1.2) we can easily obtain

$$(1.9)\qquad e_{I(p)K(n-p)}e^{I(p)J(n-p)} = p!\,\varepsilon^{J(n-p)}_{K(n-p)}.$$

The dual operator $*$ is defined by (for this see, for instance, [5])

$$(1.10)\qquad *\phi = (*\phi)_{I_0(n-p)}dx^{I_0(n-p)},$$

where

$$(1.11)\qquad (*\phi)_{I(n-p)} = \frac{1}{p!}\,e_{J(p)I(n-p)}\phi^{J(p)}.$$

From equations (1.10), (1.11) it follows that for the scalar 1

$$(1.12)\qquad *1 = (\det(g_{ij}))^{1/2}dx^1\wedge\cdots\wedge dx^n,$$

which is just the element of area of the manifold M^n. By using orthonormal local coordinates $x^1, \cdots, x^n$ we can easily verify that

$$**\phi = (-1)^{p(n-p)}\phi. \tag{1.13}$$

Denote the inverse operator of $*$ by $*^{-1}$. Then from equation (1.13) it is seen that on forms of degree p

$$*^{-1} = (-1)^{p(n-p)}* \ . \tag{1.14}$$

The codifferential operator δ is defined by

$$\delta\phi = (-1)^{p+n+1} *^{-1} d * \phi. \tag{1.15}$$

Making use of equations (1.6), (1.7), (1.10), (1.11) we obtain immediately

$$\delta\phi = (\delta\phi)_{I_0(p-1)} dx^{I_0(p-1)}, \tag{1.16}$$

where

$$(\delta\phi)_{I(p-1)} = -\nabla_j \phi^j{}_{I(p-1)}. \tag{1.17}$$

By means of equations (1.13), (1.14) it is easy to verify that

$$\Delta * = * \Delta, \tag{1.18}$$

where Δ is the Laplace-Beltrami operator defined by

$$\Delta = \delta d + d\delta. \tag{1.19}$$

For a form ϕ of degree p defined by equation (1.4) we can have

$$\begin{aligned} p!(\Delta\phi)_{I(p)} = & -\nabla^j \nabla_j \phi_{I(p)} + \sum_{s=1}^{p} \phi_{I(p;\bar{s}|a)} R^a{}_{i_s} \\ & + \sum_{s<t}^{1,\ldots,p} \phi_{I(p;\bar{s}|a,\bar{t}|b)} R^{ab}{}_{i_s i_t}, \end{aligned} \tag{1.20}$$

where

$$\nabla^j = g^{jk}\nabla_k, \tag{1.21}$$

$$R^i{}_{jkl} = \partial\Gamma^i_{jk}/\partial x^l - \partial\Gamma^i_{jl}/\partial x^k + \Gamma^h_{jk}\Gamma^i_{hl} - \Gamma^h_{jl}\Gamma^i_{hk}, \tag{1.22}$$

$$R_{jk} = R^s{}_{jks}. \tag{1.23}$$

2. **Complex structures and operators.** On a Riemannian manifold M^n with metric tensor g_{ij}, if there exists a tensor $F_i{}^j$ of type (1,1) satisfying

$$F_i{}^j F_j{}^k = -\varepsilon^k_i, \tag{2.1}$$

then $F_i{}^j$ is said to define an almost-complex structure on the manifold M^n, and the manifold M^n is called an almost-complex manifold. From equation (2.1) it follows that the almost-complex structure $F_i^{\ j}$ induces an automorphism J of the tangent space of the manifold M^n at each point with $J^2 = -I$, I being the identity operator, such that, for any tangent vector v^k,

$$J: v^k \to F_i^{\ k} v^i. \tag{2.2}$$

If an almost-complex structure $F_i^{\ j}$ further satisfies

$$g_{ij} F_h^{\ i} F_k^{\ j} = g_{hk}, \tag{2.3}$$

then $F_i^{\ j}$ is said to define an almost-Hermitian structure on the manifold M^n, and the manifold M^n is called an almost-Hermitian manifold. From equations (2.1), (2.3) it follows that the tensor F_{ij} of type (0,2) defined by

$$F_{ij} = g_{jk} F_i^{\ k} \tag{2.4}$$

is skew-symmetric. Thus on an almost-Hermitian manifold we have the associated differential form

$$\omega = F_{ij} dx^i \wedge dx^j. \tag{2.5}$$

By using the multiplication of matrices, from equation (2.1) we readily see that a necessary condition for the existence of an almost-complex structure on a Riemannian manifold M^n is that the dimension n of the manifold M^n be even. It should also be remarked that an almost-complex manifold is always orientable, and the orientation depends only on the tensor $F_i^{\ j}$.

An almost-Hermitian structure $F_i^{\ j}$ defined on a manifold M^n is called an almost-Kählerian structure and the manifold M^n an almost-Kählerian manifold, if the associated form ω is closed, that is,

$$d\omega = 0. \tag{2.6}$$

From equations (2.5), (2.6) it follows that an almost-Kählerian structure $F_i^{\ j}$ satisfies

$$F_{hij} \equiv \nabla_h F_{ij} + \nabla_i F_{jh} + \nabla_j F_{hi} = 0. \tag{2.7}$$

The tensor F_{hij} is obviously skew-symmetric in all indices.

An almost-Hermitian structure $F_i^{\ j}$ (respectively manifold) satisfying

$$F_i \equiv -\nabla_j F_i^{\ j} = 0 \tag{2.8}$$

is called an almost-semi-Kählerian structure (respectively manifold). In particular, the structure $F_i^{\ j}$ is Kählerian if $\nabla_i F_j^{\ k} = 0$. In this case, by means of equation (2.1) it is easily seen that the torsion tensor

$$t_{ij}^{\ \ k} = F_j^{\ h}(\partial F_i^{\ k}/\partial x^h - \partial F_h^{\ k}/\partial x^i) - F_i^{\ h}(\partial F_j^{\ k}/\partial x^h - \partial F_h^{\ k}/\partial x^j)$$

vanishes, so that the integrability condition of the almost-complex structure $F_i^{\ j}$

is satisfied. But in general when ${t_{ij}}^k = 0$, the almost-Hermitian structure ${F_i}^j$ is defined to be Hermitian.

Multiplying equation (2.4) by F^{hi} we obtain

$$F_{ij}F^{hi} = -\varepsilon_j^h. \tag{2.9}$$

By taking covariant differentiation of both sides of equation (2.9), noticing that

$$F^{ij}\nabla_h F_{ij} = 0, \tag{2.10}$$

and making use of equations (2.7), (2.8) it is easily seen that

$$F_{hij}F^{ij} = 2{F_h}^i F_i. \tag{2.11}$$

Thus an almost-semi-Kählerian structure ${F_i}^j$ satisfies

$$F_{hij}F^{ij} = 0. \tag{2.12}$$

Multiplication of equation (2.11) by F_k^h and a use of equation (2.9) give

$$F_k = -\frac{1}{2} F_{hij}F^{ij}{F_k}^h. \tag{2.13}$$

From equations (2.7), (2.8), (2.13) we hence reach that *an almost-Kählerian structure or manifold is also almost-semi-Kählerian.*

We now consider an almost-Hermitian manifold M^n with an almost-Hermitian structure F_i^j so that equation (2.3) holds, and shall introduce complex operators (compare [6, Chapter IX]) on the manifold M^n. At first we define

$$\underset{1,0}{\Pi}{}_i{}^j = \frac{1}{2}(g_i^j - \sqrt{(-1)}{F_i}^j) \tag{2.14}$$

and its conjugate[2] tensor

$$\underset{0,1}{\Pi}{}_i{}^j = \underset{1,0}{\overline{\Pi}}{}_i{}^j = \frac{1}{2}({g_i}^j + \sqrt{(-1)}F_i^j). \tag{2.15}$$

Let $\rho + \sigma = p$, $\rho \geqq 0$, $\sigma \geqq 0$, and set

$$\begin{aligned}\underset{\rho,\sigma}{\Pi}{}_{I(p)}{}^{J(p)} = \varepsilon_{I(p)}^{M(\rho)N(\sigma)} \underset{1,0}{\Pi}{}_{m_1}{}^{r_1} \cdots \underset{1,0}{\Pi}{}_{m_\rho}{}^{r_\rho} \\ \cdot \underset{0,1}{\Pi}{}_{n_1}{}^{s_1} \cdots \underset{0,1}{\Pi}{}_{n_\sigma}{}^{s_\sigma} \varepsilon^{J(p)}_{R_0(\rho)S_0(\sigma)}.\end{aligned} \tag{2.16}$$

Then for a form ϕ given by equation (1.4) we have

$$\underset{\rho,\sigma}{\Pi}\phi = \Big(\underset{\rho,\sigma}{\Pi}\phi\Big)_{I_0(p)} dx^{I_0(p)}, \tag{2.17}$$

(2) Throughout this paper a bar over a letter or symbol denotes the conjugate of the complex number or operator defined by the letter or symbol.

where

$$\left(\prod_{\rho,\sigma} \phi\right)_{I(p)} = \prod_{\rho,\sigma} {}_{I(p)}{}^{J_0(p)} \phi_{J_0(p)}. \tag{2.18}$$

We next define a complex covariant differentiator

$$\mathscr{D}_i = \prod_{1,0} {}_i{}^j \nabla_j, \tag{2.19}$$

and the corresponding contravariant differentiator

$$\mathscr{D}^i = g^{ik}\mathscr{D}_k = \prod_{0,1} {}_j{}^i \nabla^j = \overline{\prod_{1,0}} {}_j{}^i \nabla^{\bar{j}}, \tag{2.20}$$

where

$$\nabla^j = g^{ij} \nabla_i. \tag{2.21}$$

The conjugate operators of $\mathscr{D}_i$ and $\mathscr{D}^i$ are

$$\overline{\mathscr{D}}_i = \prod_{0,1} {}_i{}^j \nabla_j, \tag{2.22}$$

$$\overline{\mathscr{D}}^i = \prod_{1,0} {}_j{}^i \nabla^j. \tag{2.23}$$

Furthermore, for a form ϕ given by equation (1.4) we define the complex analogues of the real operators d and δ:

$$\begin{aligned} (\partial\phi)_{I(p+1)} &= \left(\sum_{\rho+\sigma=p} \prod_{\rho+1,\sigma} d \prod_{\rho,\sigma} \phi\right)_{I(p+1)} \\ &= \sum_{\rho+\sigma=p} \prod_{\rho+1,\sigma} {}_{I(p+1)}{}^{jJ_0(p)} \mathscr{D}_j \phi_{J_0(p)}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} (\mathfrak{D}\phi)_{I(p-1)} &= \left(\sum_{\rho+\sigma=p} \prod_{\rho,\sigma-1} \delta \prod_{\rho,\sigma} \phi\right)_{I(p-1)} \\ &= -\sum_{\rho+\sigma=p} \prod_{\rho,\sigma} {}_{iI(p-1)}{}^{J_0(p)} \mathscr{D}^i \phi_{J_0(p)}. \end{aligned} \tag{2.25}$$

The conjugate operators of ∂ and $\mathfrak{D}$ have the forms

$$\begin{aligned} (\bar{\partial}\phi)_{I(p+1)} &= \left(\sum_{\rho+\sigma=p} \prod_{\rho,\sigma+1} d \prod_{\rho,\sigma} \phi\right)_{I(p+1)} \\ &= \sum_{\rho+\sigma=p} \prod_{\rho,\sigma+1} {}_{I(p+1)}{}^{jJ_0(p)} \overline{\mathscr{D}}_j \phi_{J_0(p)}, \end{aligned} \tag{2.26}$$

$$(\bar\theta\phi)_{I(p-1)} = \Big(\sum_{\rho+\sigma=p} \underset{\rho-1,\sigma}{\Pi}\ \delta\ \underset{\rho,\sigma}{\Pi}\ \phi\Big)_{I(p-1)} \tag{2.27}$$
$$= -\sum_{\rho+\sigma=p} \underset{\rho,\sigma}{\Pi}{}_{iI(p-1)}{}^{J_0(p)}\bar{\mathscr{D}}^i\phi_{J_0(p)}.$$

Now we introduce a complex Laplace-Beltrami operator

$$\Box = \bar{\mathfrak{D}}\partial + \partial\bar{\mathfrak{D}} \tag{2.28}$$

and its conjugate operator

$$\bar\Box = \mathfrak{D}\bar\partial + \bar\partial\mathfrak{D}. \tag{2.29}$$

3. **Proof of Theorem 3.1.** Let M^n be an almost-Hermitian manifold with an almost-Hermitian structure $F_i^{\ j}$ so that equation (2.3) holds. Then on the manifold M^n we can define the real operators L and Λ as follows:

$$L\phi = \phi \wedge \omega \tag{3.1}$$

for any form ϕ, and

$$\Lambda = *^{-1}L*. \tag{3.2}$$

From the known fact that Δ commutes with $*$ and $*^{-1}$, it follows from equation (3.2) immediately that if Δ commutes with L, it also commutes with Λ. Therefore it is sufficient to prove the theorem under the commutativity of Δ with Λ, that is, under the condition that for all forms ϕ of any degree p $(0 \leqq p \leqq n-2)$

$$\Delta(\phi \wedge \omega) = (\Delta\phi) \wedge \omega. \tag{3.3}$$

By means of equation (1.20), (2.5) for any form ϕ of degree p $(0 \leqq p \leqq n-2)$ given by

$$\phi = \phi_{I(p)}dx^{I(p)}, \tag{3.4}$$

we can easily obtain

$$[(\Delta\phi) \wedge \omega]_{I(p)j_1j_2} = -(\nabla^j\nabla_j\phi_{I(p)})F_{j_1j_2} + \sum_{s=1}^{p} \phi_{I(p;\hat{s}|a)}R^a{}_{i_s}F_{j_1j_2} \tag{3.5}$$
$$+ \sum_{s<t}^{1,\dots,p} \phi_{I(p;\hat{s}|a,\hat{t}|b)}R^{ab}{}_{i_si_t}F_{j_1j_2},$$

$$\phi \wedge \omega = \Big(\phi_{I(p)}F_{j_1j_2} - \sum_{s=1}^{p} \phi_{I(p;\hat{s}|j_1)}F_{i_sj_2} \tag{3.6}$$
$$- \sum_{s=1}^{p} \phi_{I(p;\hat{s}|j_2)}F_{j_1i_s} + \sum_{s<t}^{1,\dots,p} \phi_{I(p;\hat{s}|j_1,\hat{t}|j_2)}F_{i_si_t}\Big)$$
$$\cdot\ dx^{I(p)} \wedge dx^{j_1} \wedge dx^{j_2},$$

$$
\begin{aligned}
[\Delta(\phi \wedge \omega)]_{I(p)j_1j_2} = & - \nabla^j\nabla_j(\phi_{I(p)}F_{j_1j_2}) \\
& + \sum_{s=1}^{p} \phi_{I(p;s|a)}F_{j_1j_2}R^a{}_{i_s} + \phi_{I(p)}F_{aj_2}R^a{}_{j_1} \\
& + \phi_{I(p)}F_{j_1a}R^a{}_{j_2} + \sum_{s<t}^{1,\dots,p} \phi_{I(p;s|a\ t|b)}F_{j_1j_2}R^{ab}{}_{i_st} \\
& + \sum_{s=1}^{p} \phi_{I(p;s|a)}F_{bj_2}R^{ab}{}_{i_sj_1} + \sum_{s=1}^{p} \phi_{I(p;s|a)}F_{j_1b}R^{ab}{}_{i_sj_2} \\
& + \phi_{I(p)}F_{ab}R^{ab}{}_{j_1j_2} - \sum_{s=1}^{p} \phi_{I(p;s|a)}F_{i\ j_2}R^a{}_{j_1} \\
& - \sum_{q<s}^{1,\dots,p} \phi_{I(p;q|a\ s|b)}F_{i\ j_2}R^{ab}{}_{i\ j_1} \\
& - \sum_{s<t}^{1,\dots,p} \phi_{I(p;s|a\ t|b)}F_{i_sj_2}R^{ab}{}_{j_1} \\
(3.7) \qquad & - \sum_{s=1}^{p} \phi_{I(p;s|a)}(F_{bj_2}R^{ab}{}_{j_1i_s} + F_{i_sb}R^{ab}{}_{j_1j_2}) \\
& - \sum_{s=1} \phi_{I(p;s|a)}F_{j_1i_s}R^a{}_{j_2} \\
& - \sum_{q<s}^{1,\dots,p} \phi_{I(p;q|a.s|b)}F_{j_1i_s}R^{ab}{}_{i_qj_2} \\
& - \sum_{s<t}^{1,\dots,p} \phi_{I(p;s|a,t|b)}F_{j_1i_s}R^{ab}{}_{j_2i_t} \\
& - \sum_{s=1}^{p} \phi_{I(p;s|a)}(F_{j_1b}R^{ab}{}_{j_2i} + F_{bi}R^{ab}{}_{j_2j_1}) \\
& + \sum_{s<t}^{1,\dots,p} \phi_{I(p;s|a\ t|b)}F_{i_si_t}R^{ab}{}_{j_1j_2} + T,
\end{aligned}
$$

where T denotes the remaining terms.

In particular, for a form ϕ, all of whose components are zero except $\phi_{I(p)} = 1$ for an arbitrary fixed set $I(p)$, we have $T = 0$. From equations (3.5), (3.7) it thus follows that equation (3.3) implies

$$(3.8) \qquad \nabla^j\nabla_jF_{j_1j_2} = G_{\ j_1j_2},$$

where we have placed

$$
\begin{aligned}
G_{j_1j_2} = {} & F_{aj_2}R^a{}_{j_1} + F_{j_1a}R^a{}_{j_2} + \sum_{s<t}^{1,\dots,p} F_{j_1j_2}R^{i_si_t}{}_{i_si_t} \\
& + 2\sum_{s=1}^{p} F_{bj_2}R^{i_sb}{}_{i_sj_1} + 2\sum_{s=1}^{p} F_{j_1b}R^{i_sb}{}_{i_sj_2} \\
& + F_{ab}R^{ab}{}_{j_1j_2} - \sum_{s=1}^{p} F_{i_sj_2}R^{i_s}{}_{j_1} \\
& - \sum_{q<s}^{1,\dots,p} F_{i_sj_2}R^{i_qi_s}{}_{i_qj_1} - \sum_{s<t}^{1,\dots,p} F_{i_sj_2}R^{i_si_t}{}_{j_1i_t} \\
& - 2\sum_{s=1}^{p} F_{i_sb}R^{i_sb}{}_{j_1j_2} - \sum_{s=1}^{p} F_{j_1i_s}R^{i_s}{}_{j_2} \\
& - \sum_{q<s}^{1,\dots,p} F_{j_1i_s}R^{i_qi_s}{}_{i_qj_2} - \sum_{s<t}^{1,\dots,p} F_{j_1i_s}R^{i_si_t}{}_{j_2i_t} \\
& + \sum_{s<t}^{1,\dots,p} F_{i_si_t}R^{i_si_t}{}_{j_1j_2}.
\end{aligned} \tag{3.9}
$$

Similarly, for a form ϕ, all of whose components are zero except $\phi_{I(p)} = x^k$ for an arbitrary fixed k $(1 \leqq k \leqq n)$ and set $I(p)$, we also have $T = 0$. Now at a general point P of the manifold M^n let us choose orthogonal geodesic local coordinates $x^1, \cdots, x^n$ so that

$$g_{ij}(P) = \delta_{ij}, \qquad \Gamma^k_{ij}(P) = 0, \tag{3.10}$$

where δ_{ij} are Kronecker deltas. Then for this form ϕ, at the point P equations (3.5), (3.7) are reduced to

$$[(\Delta\phi) \wedge \omega]_{I(p)j_1j_2} = x^k \sum_{s=1}^{p} F_{j_1j_2}R^{i_s}{}_{i_s}, \tag{3.11}$$

$$
\begin{aligned}
[\Delta(\phi \wedge \omega)]_{I(p)j_1j_2} = {} & -2\nabla_k F_{j_1j_2} - x^k\nabla^j\nabla_j F_{j_1j_2} \\
& + x^k \sum_{s=1}^{p} F_{j_1j_2}R^{i_s}{}_{i_s} + x^k G_{j_1j_2}.
\end{aligned} \tag{3.12}
$$

From equations (3.11), (3.12), (3.9), (3.3), it follows immediately that $\nabla_k F_{j_1j_2} = 0$ for all k, j_1, j_2 at the point P, and hence the theorem is proved.

4. **Realization of the complex operator $\square$.** Throughout this section the complex operator $\square$ is defined with respect to an almost-Hermitian structure $F_i{}^j$ on an almost-Hermitian manifold M^n unless stated otherwise. As was mentioned in the introduction, if the structure $F_i{}^j$ is Kählerian, then the complex operator $\square$ is real and equal to $\frac{1}{2}\Delta$. In this section we shall study the converse of this relationship between the operators $\square$, Δ and the structure $F_i{}^j$.

At first, we apply the operator $\square$ to any form ξ of degree zero. From equations (2.28), (2.24), (2.27), (2.16), (2.19), (2.22), (2.14), (2.15), (1.20) it follows that

$$
\begin{aligned}
2\,\square\,\xi &= -2 \sum_{\rho+\sigma=1} \underset{\rho,\sigma}{\Pi}{}_i{}^j \bar{\mathscr{D}}^i \underset{1,0}{\Pi}{}_j{}^k \mathscr{D}_k \xi \\
&= \Delta\xi + \frac{1}{2}\nabla^h F_h{}^j(-F_j{}^k\nabla_k\xi + \sqrt{(-1)}\nabla_j\xi),
\end{aligned} \tag{4.1}
$$

since $\nabla_h\nabla_j\xi = \nabla_j\nabla_h\xi$, which implies that $F_h^j\,\nabla^h\nabla_j\xi = 0$. Thus, with respect to every ξ, $\square$ is real (and therefore $\square = \frac{1}{2}\Delta$), if and only if $\nabla^h F_h{}^j \nabla_j \xi = 0$ for every ξ, or equivalently if and only if $\nabla^h F_h{}^k = 0$ by choosing ξ to be x^k for an arbitrary k. Hence we obtain Theorem 4.1 in consequence of equation (2.8).

We next apply the operator $\square$ to any form η of degree one. From equations (2.24), $\cdots$, (2.28), (2.16) it follows that

$$
(\square\eta)_{i_1} = -(\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_2)\bar{\mathscr{D}}^i(\mathrm{II}_1 + \mathrm{II}_2)\mathscr{D}_k\eta_{k_1} - \mathrm{III}, \tag{4.2}
$$

where we have placed

$$
\begin{aligned}
\mathrm{I}_1 &= \varepsilon^{m_1 n_1}_{i i_1} \underset{1,0}{\Pi}{}_{m_1}{}^{r_1} \underset{0,1}{\Pi}{}_{n_1}{}^{s_1} \varepsilon^{(j_1 j_2)}_{r_1 s_1}, \\
\mathrm{I}_2 &= \varepsilon^{m_1 m_2}_{i i_1} \underset{1,0}{\Pi}{}_{m_1}{}^{r_1} \underset{1,0}{\Pi}{}_{m_2}{}^{r_2} \varepsilon^{(j_1 j_2)}_{(r_1 r_2)}; \\
\mathrm{II}_1 &= \varepsilon^{a_1 a_2}_{(j_1 j_2)} \underset{1,0}{\Pi}{}_{a_1}{}^{u_1} \underset{1,0}{\Pi}{}_{a_2}{}^{u_2} \varepsilon^{k k_1}_{(u_1 u_2)}, \\
\mathrm{II}_2 &= \varepsilon^{a_1 b_1}_{(j_1 j_2)} \underset{1,0}{\Pi}{}_{a_1}{}^{u_1} \underset{0,1}{\Pi}{}_{b_1}{}^{v_1} \varepsilon^{k k_1}_{u_1 v_1}; \\
\mathrm{III} &= \underset{1,0}{\Pi}{}_{i_1}{}^j \mathscr{D}_j \left(\underset{1,0}{\Pi}{}_i{}^{j_1} + \underset{0,1}{\Pi}{}_i{}^{j_1} \right) \bar{\mathscr{D}}^i \eta_{j_1},
\end{aligned} \tag{4.3}
$$

$(j_1 \cdots j_p)$ indicating that the indices $j_1, \cdots, j_p$ are in the natural order. By means of equations (2.14), (2.15), (4.3) an elementary calculation yields

$$
\begin{aligned}
\mathrm{I}_1 &= \underset{1,0}{\Pi}{}_i{}^{j_1} \underset{0,1}{\Pi}{}_{i_1}{}^{j_2} - \underset{1,0}{\Pi}{}_{i_1}{}^{j_1} \underset{0,1}{\Pi}{}_i{}^{j_2} \\
&\qquad - \underset{1,0}{\Pi}{}_i{}^{j_2} \underset{0,1}{\Pi}{}_{i_1}{}^{j_1} + \underset{0,1}{\Pi}{}_{i_1}{}^{j_2} \underset{0,1}{\Pi}{}_i{}^{j_1} \quad (j_1 < j_2) \\
&= \frac{1}{2}(g_i{}^{j_1} g_{i_1}{}^{j_2} - g_{i_1}{}^{j_1} g_i{}^{j_2} + F_i{}^{j_1} F_{i_1}{}^{j_2} - F_{i_1}{}^{j_1} F_i{}^{j_2}) \quad (j_1 < j_2),
\end{aligned} \tag{4.4a}
$$

$$
\begin{aligned}
\mathrm{I}_2 &= \underset{1,0}{\Pi}{}_i{}^{j_1} \underset{1,0}{\Pi}{}_{i_1}{}^{j_2} - \underset{1,0}{\Pi}{}_{i_1}{}^{j_1} \underset{1,0}{\Pi}{}_i{}^{j_2} \quad (j_1 < j_2) \\
&= \frac{1}{4}[g_i{}^{j_1} g_{i_1}{}^{j_2} - g_{i_1}{}^{j_1} g_i{}^{j_2} + F_{i_1}{}^{j_1} F_i{}^{j_2} - F_i{}^{j_1} F_{i_1}{}^{j_2} \\
&\quad + \sqrt{(-1)}(g_{i_1}{}^{j_1} F_i{}^{j_2} + g_i{}^{j_2} F_{i_1}{}^{j_1} - g_i{}^{j_1} F_{i_1}{}^{j_2} - g_{i_1}{}^{j_2} F_i{}^{j_1})] \quad (j_1 < j_2),
\end{aligned}
$$

$$\mathrm{II}_1 = \underset{1,0}{\Pi}{}_{j_1}{}^{k} \underset{1,0}{\Pi}{}_{j_2}{}^{k_1} - \underset{1,0}{\Pi}{}_{j_2}{}^{k} \underset{1,0}{\Pi}{}_{j_1}{}^{k_1} \quad (j_1 < j_2)$$

$$= \frac{1}{4}\,[g_{j_1}{}^{k}g_{j_2}{}^{k_1} - g_{j_2}{}^{k}g_{j_1}{}^{k_1} + F_{j_2}{}^{k}F_{j_1}{}^{k_1} - F_{j_1}{}^{k}F_{j_2}{}^{k_1}$$

$$(4.4b) \qquad + \sqrt{(-1)}(g_{j_2}{}^{k}F_{j_1}{}^{k_1} + g_{j_1}{}^{k_1}F_{j_2}{}^{k} - g_{j_1}{}^{k}F_{j_2}{}^{k_1} - g_{j_2}{}^{k_1}F_{j_1}{}^{k})]\ (j_1 < j_2),$$

$$\mathrm{II}_2 = \frac{1}{2}\,(g_{j_1}{}^{k}g_{j_2}{}^{k_1} - g_{j_2}{}^{k}g_{j_1}{}^{k_1} + F_{j_1}{}^{k}F_{j_2}{}^{k_1} - F_{j_2}{}^{k}F_{j_1}{}^{k_1}) \quad (j_1 < j_2),$$

$$4\mathrm{III} = \nabla_{i_1}\nabla^{j}\eta_j - F_{i_1}{}^{h}F_{i}{}^{j}\nabla_h\nabla^{i}\eta_j - F_{i_1}{}^{i}\nabla_i F_h{}^{j}\nabla^h\eta_j$$
$$- \sqrt{(-1)}(F_{i_1}{}^{k}\nabla_k\nabla^j\eta_j + \nabla_{i_1}F_k{}^{j}\nabla^k\eta_j + F_k{}^{j}\nabla_{i_1}\nabla^k\eta_j).$$

From equations (4.4a,b), (2.19), (2.22) we thus obtain

$$(\mathrm{I}_1 + \mathrm{I}_2 + \bar{\mathrm{I}}_2)\bar{\mathscr{D}}^i(\mathrm{II}_1 + \mathrm{II}_2)\mathscr{D}_k\eta_{k_1} = \frac{1}{2}\,(g_i{}^{j_1}g_{i_1}{}^{j_2} - g_{i_1}{}^{j_1}g_i{}^{j_2})$$
$$\cdot\,\bar{\mathscr{D}}^i[g_{j_1}{}^{h}g_{j_2}{}^{k_1} - g_{j_2}{}^{h}g_{j_1}{}^{k_1} + \sqrt{(-1)}(g_{j_1}{}^{k_1}F_{j_2}{}^{h} - g_{j_2}{}^{k_1}F_{j_1}{}^{h})]\cdot\nabla_h\eta_{k_1}\,(j_1 < j_2)$$
$$= \frac{1}{2}\,[(g_i{}^{j_1}g_{i_1}{}^{j_2} - g_{i_1}{}^{j_1}g_i{}^{j_2})\bar{\mathscr{D}}^i(g_{j_1}{}^{h}g_{j_2}{}^{k_1} + \sqrt{(-1)}g_{j_1}{}^{k_1}F_{j_2}{}^{h})\nabla_h\eta_{k_1}]$$
$$= \frac{1}{4}\,[2\nabla^i\nabla_i\eta_{i_1} - \nabla^j\nabla_{i_1}\eta_j + F_j{}^{h}\nabla^i F_i{}^{j}\nabla_h\eta_{i_1} + F_{i_1}{}^{h}F_i{}^{j}\nabla^i\nabla_h\eta_j$$
$$+ F_i{}^{j}\nabla^i F_{i_1}{}^{h}\nabla_h\eta_j + \sqrt{(-1)}(F_{i_1}{}^{k}\nabla^j\nabla_k\eta_j - 2F_j{}^{k}\nabla^j\nabla_k\eta_{i_1}$$
$$+ \nabla^j F_{i_1}{}^{k}\nabla_k\eta_j - \nabla^i F_i{}^{j}\nabla_j\eta_{i_1} + F_k{}^{j}\nabla^k\nabla_{i_1}\eta_j)],$$

which with equations (4.2), (4.4) give immediately

$$4(\Box\eta)_{i_1} = -2\nabla^i\nabla_i\eta_{i_1} + [\nabla_j, \nabla_{i_1}]\eta^j - F_j{}^{h}\nabla^i F_i{}^{j}\nabla_h\eta_{i_1}$$
$$+ F_{i_1}{}^{h}F^{ij}[\nabla_h, \nabla_i]\eta_j - F_i{}^{j}\nabla^i F_{i_1}{}^{h}\nabla_h\eta_j$$
$$(4.5) \qquad + F_{i_1}{}^{i}\nabla_i F_h{}^{j}\nabla^h\eta_j + \sqrt{(-1)}\{\nabla^i F_i{}^{j}\nabla_j\eta_{i_1}$$
$$- (\nabla_j F_{i_1}{}^{k} + \nabla_{i_1}F_j{}^{k})\nabla_k\eta^j + 2F_j{}^{k}\nabla^j\nabla_k\eta_{i_1}$$
$$- F_{i_1}{}^{k}[\nabla_j, \nabla_k]\eta^j - F^{kj}[\nabla_k, \nabla_{i_1}]\eta_j\},$$

where

$$(4.6) \qquad [\nabla_h, \nabla_i] = \nabla_h\nabla_i - \nabla_i\nabla_h.$$

For the proofs of Theorem 4.2 and 4.3, we need the following

LEMMA 4.1. *If the complex operator $\Box$ for an almost-Hermitian structure on an almost-Hermitian manifold M^n is real with respect to all forms of degrees* 0 *and* 1, *then the structure tensor $F_i{}^j$ satisfies*

(4.7) $$\nabla_i F_j{}^k + \nabla_j F_i{}^k = 0,$$

(4.8) $$F_i{}^h R^i{}_{jkl} = F_j{}^i R^h{}_{ikl}.$$

Proof. We first choose orthogonal geodesic local coordinates $x^1, \cdots, x^n$ at a general point P of the manifold M^n, so that equations (3.10) hold. Then at the point P for the form

(4.9) $$\eta = x^h dx^i \quad \text{for any fixed distinct } h \text{ and } i,$$

all the second covariant derivatives of any of its components being vanishing, Theorem 4.1 and the assumption that Im $(\Box\eta) = 0$, Im denoting the imaginary part, imply immediately equation (4.7). Thus for general local coordinates $x^1, \cdots, x^n$ and a general form η of degree 1, from equation (4.5) we can reduce the condition Im $(\Box\eta)_{i_1} = 0$ to

(4.10) $$2F_j{}^k \nabla^j \nabla_k \eta_{i_1} - F_{i_1}{}^k [\nabla_j, \nabla_k]\eta^j - F^{kj}[\nabla_k, \nabla_{i_1}]\eta_j = 0.$$

By using the Ricci identity for any tensor $\phi_{J(q)}{}^{I(p)}$ of type (p, q)

(4.11) $$[\nabla_l, \nabla_k]\phi_{J(q)}{}^{I(p)} = \sum_{s=1}^{p} \phi_{J(q)}{}^{I(p;s|a)} R^{i_s}{}_{akl} - \sum_{t=1}^{q} \phi_{J(q;t|b)}{}^{I(p)} R^b{}_{j_t kl},$$

and the Bianchi identity

(4.12) $$R^h{}_{ijk} + R^h{}_{jki} + R^h{}_{kij} = 0,$$

and noticing that

$$2F_j{}^k \nabla^j \nabla_k \eta_{i_1} = F^{jk}[\nabla_j, \nabla_k]\eta_{i_1},$$

an elementary calculation from equation (4.10) gives readily equation (4.8). Hence the lemma is proved.

It should be noted that due to equation (4.7) F_{ij} is a Killing tensor (for this see, for instance, [8]), and it is well known that on the 6-sphere S^6 there exists an almost-Hermitian structure [2], whose tensor is a Killing tensor [3]. It is also well known that the relation (4.8) holds automatically for a Kählerian structure $F_i{}^j$. In fact, by means of the identity (4.11) we have

(4.13) $$0 = [\nabla_l, \nabla_k]F_i{}^j = F_i{}^a R^j{}_{akl} - F_b{}^j R^b{}_{ikl}.$$

Multiplication of equation (4.13) by $g^{ih} g_{jm}$ leads immediately to equation (4.8).

For the proof of Theorem 4.2, we first use the identity (4.11) and equations (4.8), (2.1), (4.7) to obtain

(4.14) $$\begin{aligned} [\nabla_j, \nabla_{i_1}]\eta^j &= \eta_a R^a{}_{i_1} = F_{i_1}{}^h F^{ij}[\nabla_h, \nabla_i]\eta_j, \\ &\quad - F_i{}^j \nabla^i F_{i_1}{}^h \nabla_h \eta_j + F_{i_1}{}^i \nabla_i F_h{}^j \nabla^h \eta_j \\ &= F_i{}^j \nabla_h \eta_j (\nabla^h F_{i_1}{}^i - \nabla^i F_{i_1}{}^h) = 2F_i{}^j \nabla^h F_{i_1}{}^i \nabla_h \eta_j. \end{aligned}$$

Then from equations (4.5), (1.20), (4.13) and Theorem 4.1 it follows that

$$2(\Box\eta)_{i_1} = (\Delta\eta)_{i_1} + F_i{}^j \nabla^h F_{i_1}{}^i \nabla_h \eta_j. \tag{4.15}$$

As before, by choosing orthogonal geodesic local coordinates $x^1, \cdots, x^n$ at a general point P of the manifold M^n and considering the form η given by equation (4.9), from equation (4.15) we readily see that the condition $\Box = \frac{1}{2}\Delta$ implies that

$$F_j{}^i \nabla^h F_{i_1}{}^j = 0 \quad \text{for each } i_1, i, h. \tag{4.16}$$

Multiplication of equation (4.16) by $F_i{}^k$ and use of equation (2.1) thus give $\nabla^h F_{i_1}{}^k = 0$ for each i_1, h, k, which completes the proof of Theorem 4.2.

Finally we apply the operator $\Box$ to any form ζ of degree two. From equations (2.24), $\cdots$, (2.28), (2.16) it follows that

$$\begin{aligned}(\Box\zeta)_{i_1 i_2} = &-(\mathrm{IV}_1 + \mathrm{IV}_2 + \overline{\mathrm{IV}}_1 + \overline{\mathrm{IV}}_2)\, \overline{\mathscr{D}}^i (\mathrm{V}_1 + \mathrm{V}_2 + \overline{\mathrm{V}}_2)\, \mathscr{D}_k \zeta_{(k_1 k_2)} \\ &- (\mathrm{VI}_1 + \mathrm{VI}_2) \mathscr{D}_j (\mathrm{VII}_1 + \mathrm{VII}_2 + \overline{\mathrm{VII}}_1) \overline{\mathscr{D}})^i \zeta_{(k_1 k_2)},\end{aligned} \tag{4.17}$$

where we have placed

$$\begin{aligned}\mathrm{IV}_1 &= \varepsilon_{i i_1 i_2}^{m_1 m_2 m_3} \prod_{1,0} {}_{m_1}{}^{r_1} \prod_{1,0} {}_{m_2}{}^{r_2} \prod_{1,0} {}_{m_3}{}^{r_3} \varepsilon_{(r_1 r_2 r_3)}^{(j_1 j_2 j_3)}, \\ \mathrm{IV}_2 &= \varepsilon_{i i_1 i_2}^{m_1 m_2 n_1} \prod_{1,0} {}_{m_1}{}^{r_1} \prod_{1,0} {}_{m_2}{}^{r_2} \prod_{0,1} {}_{n_1}{}^{s_1} \varepsilon_{(r_1 r_2) s_1}^{(j_1 j_2 j_3)};\end{aligned} \tag{4.18}$$

$$\begin{aligned}\mathrm{V}_1 &= \varepsilon_{(j_1 j_2 j_3)}^{a_1 a_2 a_3} \prod_{1,0} {}_{a_1}{}^{u_1} \prod_{1,0} {}_{a_2}{}^{u_2} \prod_{1,0} {}_{a_3}{}^{u_3} \varepsilon_{(u_1 u_2 u_3)}^{k(k_1 k_2)}, \\ \mathrm{V}_2 &= \varepsilon_{(j_1 j_2 j_3)}^{a_1 a_2 b_1} \prod_{1,0} {}_{a_1}{}^{u_1} \prod_{1,0} {}_{a_2}{}^{u_2} \prod_{0,1} {}_{b_1}{}^{v_1} \varepsilon_{(u_1 u_2) v_1}^{k(k_1 k_2)};\end{aligned} \tag{4.19}$$

$$\begin{aligned}\mathrm{VI}_1 &= \varepsilon_{i_1 i_2}^{m_1 m_2} \prod_{1,0} {}_{m_1}{}^{r_1} \prod_{1,0} {}_{m_2}{}^{r_2} \varepsilon_{(r_1 r_2)}^{j j_1}, \\ \mathrm{VI}_2 &= \varepsilon_{i_1 i_2}^{m_1 n_1} \prod_{1,0} {}_{m_1}{}^{r_1} \prod_{0,1} {}_{n_1}{}^{s_1} \varepsilon_{r_1 s_1}^{j j_1};\end{aligned} \tag{4.20}$$

$$\begin{aligned}\mathrm{VII}_1 &= \varepsilon_{i j_1}^{a_1 a_2} \prod_{1,0} {}_{a_1}{}^{u_1} \prod_{1,0} {}_{a_2}{}^{u_2} \varepsilon_{(u_1 u_2)}^{(k_1 k_2)}, \\ \mathrm{VII}_2 &= \varepsilon_{i j_1}^{a_1 b_1} \prod_{1,0} {}_{a_1}{}^{u_1} \prod_{0,1} {}_{b_1}{}^{v_1} \varepsilon_{u_1 v_1}^{(k_1 k_2)}.\end{aligned} \tag{4.21}$$

By means of equations (2.14), (2.15), (4.18) an elementary but rather lengthy calculation gives

$$\mathrm{IV}_1 + \mathrm{IV}_2 + \overline{\mathrm{IV}}_1 + \overline{\mathrm{IV}}_2 = \lambda_{i i_1 i_2}^{(j_1 j_2 j_3)}, \tag{4.22}$$

where $\lambda_{i i_1 i_2}^{j_1 j_2 j_3}$ is skew-symmetric in j_1, j_2, j_3 and given by

$$\begin{aligned}\lambda_{i i_1 i_2}^{j_1 j_2 j_3} = &\, g_{i_1}{}^{j_2} g_{i_2}{}^{j_3} g_i{}^{j_1} - g_i{}^{j_1} g_{i_2}{}^{j_2} g_{i_1}{}^{j_3} - g_{i_1}{}^{j_1} g_i{}^{j_2} g_{i_2}{}^{j_3} \\ &+ g_{i_2}{}^{j_1} g_i{}^{j_2} g_{i_1}{}^{j_3} + g_{i_1}{}^{j_1} g_{i_2}{}^{j_2} g_i{}^{j_3} - g_{i_2}{}^{j_1} g_{i_1}{}^{j_2} g_i{}^{j_3}.\end{aligned} \tag{4.23}$$

From equations (4.19), (4.22) it follows that

$$(4.24)\quad (\mathrm{IV}_1+\mathrm{IV}_2+\overline{\mathrm{IV}}_1+\overline{\mathrm{IV}}_2)\bar{\mathscr{D}}^i(\mathrm{V}_1+\mathrm{V}_2+\overline{\mathrm{V}}_2)=\lambda^{j_1j_2j_3}_{ii_1i_2}\bar{\mathscr{D}}^i(A_1+A_2+\overline{A_2}),$$

where

$$(4.25)\quad \begin{aligned} A_1 &= \underset{1,0}{\Pi}{}_{j_1}{}^{k}\ \underset{1,0}{\Pi}{}_{j_2}{}^{k_1}\ \underset{1,0}{\Pi}{}_{j_3}{}^{k_2} \qquad (k_1<k_2),\\ A_2 &= \underset{1,0}{\Pi}{}_{j_1}{}^{k}\ \underset{1,0}{\Pi}{}_{j_2}{}^{k_1}\ \underset{0,1}{\Pi}{}_{j_3}{}^{k_2} - \underset{1,0}{\Pi}{}_{j_1}{}^{k}\ \underset{1,0}{\Pi}{}_{j_2}{}^{k_2}\ \underset{0,1}{\Pi}{}_{j_3}{}^{k_1}\\ &\quad + \underset{1,0}{\Pi}{}_{j_1}{}^{k_1}\ \underset{1,0}{\Pi}{}_{j_2}{}^{k_2}\ \underset{0,1}{\Pi}{}_{j_3}{}^{k} \qquad (k_1<k_2). \end{aligned}$$

Substituting equations (2.14), (2.15) in equations (4.25) we can easily obtain

$$(4.26)\quad \begin{aligned} 8\lambda^{j_1j_2j_3}_{ii_1i_2}(A_1+A_2+\bar{A}_2) &= \lambda^{j_1j_2j_3}_{ii_1i_2}[7g_{j_1}{}^{k}g_{j_2}{}^{k_1}g_{j_3}{}^{k_2}\\ &\quad + g_{j_1}{}^{k}F_{j_2}{}^{k_1}F_{j_3}{}^{k_2} + g_{j_2}{}^{k_1}F_{j_1}{}^{k}F_{j_3}{}^{k_2} + g_{j_3}{}^{k_2}F_{j_1}{}^{k}F_{j_2}{}^{k_1}\\ &\quad + \sqrt{(-1)}(F_{j_1}{}^{k}F_{j_2}{}^{k_1}F_{j_3}{}^{k_2} - g_{j_1}{}^{k}g_{j_2}{}^{k_1}F_{j_3}{}^{k_2}\\ &\quad - g_{j_1}{}^{k}g_{j_3}{}^{k_2}F_{j_2}{}^{k_1} - g_{j_2}{}^{k_1}g_{j_3}{}^{k_2}F_{j_1}{}^{k})] \qquad (k_1<k_2), \end{aligned}$$

and therefore

$$(4.27)\quad \begin{aligned} &\lambda^{j_1j_2j_3}_{ii_1i_2}(A_1+A_2+\bar{A}_2)\ \underset{1,0}{\Pi}{}_k{}^h\\ &= \frac{1}{2}\lambda^{j_1j_2j_3}_{ii_1i_2}(g_{j_1}{}^{h}g_{j_2}{}^{k_1}g_{j_3}{}^{k_2} - \sqrt{(-1)}g_{j_2}{}^{k_1}g_{j_3}{}^{k_2}F_{j_1}{}^{h}) \qquad (k_1<k_2). \end{aligned}$$

Making use of equations (2.19), (2.22), (4.23), (4.24), (4.27) we thus have

$$(4.28)\quad \begin{aligned} &4\ \mathrm{Im}[(\mathrm{IV}_1+\mathrm{IV}_2+\overline{\mathrm{IV}}_1+\overline{\mathrm{IV}}_2)\bar{\mathscr{D}}^i(\mathrm{V}_1+\mathrm{V}_2+\overline{\mathrm{V}}_2)\mathscr{D}_k\,\zeta_{(k_1k_2)}]\\ &= -\ 2F_j{}^h\nabla^j\nabla_h\zeta_{i_1i_2} + F_j{}^k\nabla^j\nabla_{i_1}\zeta_{ki_2} - F_j{}^k\nabla^j\nabla_{i_2}\zeta_{ki_1}\\ &\quad -\ \nabla^jF_j{}^h\nabla_h\zeta_{i_1i_2} + \nabla^iF_{i_1}{}^h\nabla_h\zeta_{ii_2} - \nabla^iF_{i_2}{}^h\nabla_h\zeta_{ii_1}\\ &\quad +\ F_{i_1}{}^k\nabla^h\nabla_k\zeta_{hi_2} - F_{i_2}{}^h\nabla^i\nabla_h\zeta_{ii_1}. \end{aligned}$$

Similarly, by means of equations (4.20), (4.21), (4.3), (4.4), (2.19), (2.22) we obtain

$$(4.29)\quad \begin{aligned} (\mathrm{VI}_1+\mathrm{VI}_2)\mathscr{D}_j &= \frac{1}{2}\,[g_{i_1}{}^{k}g_{i_2}{}^{j_1} - g_{i_2}{}^{k}g_{i_1}{}^{j_1}\\ &\quad + \sqrt{(-1)}(g_{i_1}{}^{j_1}F_{i_2}{}^{k} - g_{i_2}{}^{j_1}F_{i_1}{}^{k})]\nabla_k, \end{aligned}$$

$$(4.30)\quad \begin{aligned} \nabla_k[(\mathrm{VII}_1+\mathrm{VII}_2+\overline{\mathrm{VII}}_1)\bar{\mathscr{D}}^i\zeta_{(k_1k_2)}] &= \nabla_k[(g_i^{k_1}g_{j_1}{}^{k_2} - g_{j_1}{}^{k_1}g_i^{k_2})\bar{\mathscr{D}}^i\zeta_{(k_1k_2)}]\\ &= \frac{1}{2}\,[\nabla_k\nabla^h\zeta_{hj_1} - \sqrt{(-1)}(F_h{}^i\nabla_k\nabla^h\zeta_{ij_1} + \nabla_kF_h{}^i\nabla^h\zeta_{ij_1})], \end{aligned}$$

from which it follows that

$$
\begin{aligned}
&4\ \mathrm{Im}[(\mathrm{VI}_1 + \mathrm{VI}_2)\mathscr{D}_j(\mathrm{VII}_1 + \mathrm{VII}_2 + \overline{\mathrm{VII}}_1)\bar{\mathscr{D}}^i\zeta_{(k_1k_2)}] \\
(4.31)\qquad &= - F_j{}^k\nabla_{i_1}\nabla^j\zeta_{ki_2} - \nabla_{i_1}F_h{}^i\nabla^h\zeta_{ii_2} + F_j{}^k\nabla_{i_2}\nabla^j\zeta_{ki_1} \\
&\quad + \nabla_{i_2}F_h{}^i\nabla^h\zeta_{ii_1} + F_{i_2}{}^h\nabla_h\nabla^i\zeta_{ii_1} - F_{i_1}{}^k\nabla_k\nabla^h\zeta_{hi_2}.
\end{aligned}
$$

Substitution of equations (4.28), (4.31) in equation (4.17) thus yields

$$
\begin{aligned}
4\ \mathrm{Im}(\Box\zeta)_{i_1i_2} &= \nabla^jF_j{}^h\nabla_h\zeta_{i_1i_2} - (\nabla_{i_1}F_i{}^h + \nabla_iF_{i_1}{}^h)\nabla_h\zeta^i{}_{i_2} \\
(4.32)\qquad &\quad + (\nabla_iF_{i_2}{}^h + \nabla_{i_2}F_i{}^h)\nabla_h\zeta^i{}_{i_1} + 2F_j{}^h\nabla^j\nabla_h\zeta_{i_1i_2} \\
&\quad + F^{jk}[\nabla_{i_1}, \nabla_j]\zeta_{ki_2} + F^{jk}[\nabla_j, \nabla_{i_2}]\zeta_{ki_1} \\
&\quad + g^{hj}F_{i_1}{}^k[\nabla_k, \nabla_j]\zeta_{hi_2} + g^{ij}F_{i_2}{}^h[\nabla_j, \nabla_h]\zeta_{ii_1}.
\end{aligned}
$$

On the other hand, by using equation (4.8) and the identities (4.11), (4.12) we can obtain

$$
\begin{aligned}
2F_j{}^h\nabla^j\nabla_h\zeta_{i_1i_2} &= F^{jh}[\nabla_j, \nabla_h]\zeta_{i_1i_2} \\
&= 2F_{i_1}{}^jR_j{}^h\zeta_{hi_2} + 2F_{i_2}{}^jR_j{}^h\zeta_{i_1h}, \\
F^{jk}[\nabla_{i_1}, \nabla_j]\zeta_{ki_2} &= - F_{i_1}{}^jR_j{}^h\zeta_{hi_2} - F^{jk}\zeta_{kl}R^l{}_{i_2ji_1}, \\
(4.33)\qquad F^{jk}\zeta_{kl}R^l{}_{ji_1i_2} &= F^{jl}\zeta_{lk}R^k{}_{ji_1i_2} = - F^{jk}\zeta_{kl}R^l{}_{ji_1i_2} = 0, \\
F^{jk}[\nabla_j, \nabla_{i_2}]\zeta_{ki_1} &= - F_{i_2}{}^jR_j{}^h\zeta_{i_1h} + F^{jk}\zeta_{kl}R^l{}_{i_2ji_1}, \\
g^{hj}F_{i_1}{}^k[\nabla_k, \nabla_j]\zeta_{hi_2} &= - F_{i_1}{}^jR_j{}^h\zeta_{hi_2} - F_k{}^j\zeta_j{}^lR^k{}_{i_1li_2}, \\
g^{ij}F_{i_2}{}^h[\nabla_j, \nabla_h]\zeta_{ii_1} &= - F_{i_2}{}^jR_j{}^h\zeta_{i_1h} + F_h{}^i\zeta_i{}^lR^h{}_{i_1li_2}.
\end{aligned}
$$

Theorem 4.1 and equations (4.7), (4.33) thus reduce the right side of equation (4.32) to zero, and hence Theorem 4.3 is proved.

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